6. Three different positive integers have the property that each of them divides the sum of the other two.

Find all such sets of three numbers.

*Solution*

Let the integers be $a$, $b$, $c$, where $0 < a < b < c$. Hence $a + b < 2c$, but $c$ is a factor of $a + b$, so $a + b = c$.

Thus the integers are $a$, $b$ and $a + b$.

Now we also require $b$ to divide $2a + b$. It follows that $b$ divides $2a$. But $2a < 2b$ so $b = 2a$, and therefore $c = 3a$. But then the condition that $a$ divides $b + c$ automatically holds.

Hence there is an infinite number of solutions: $(a, b, c) = (k, 2k, 3k)$, where $k$ is any positive integer.

*Note*: For any $n \geqslant 3$, there exists a list of $n$ different positive integers such that each of them divides the sum of the remaining $n - 1$ numbers. However, classifying all solutions for general $n$ is an unsolved problem. One possible list is: $1, 2, 3 \times 2^0, 3 \times 2^1, \dots, 3 \times 2^{n-3}$. Can you prove that this list has the desired property?

This booklet contains the questions and solutions for the follow-up competitions of the UKMT Intermediate Mathematical Challenge held in February 2012.

For the age ranges covered, see the details on the relevant paper.

As is usual, it is not intended that these solutions should be thought of as the ‘best’ possible solutions and the ideas of readers may be equally meritorious.

## Contents

**EUROPEAN 'KANGAROO' MATHEMATICAL CHALLENGE**
**'GREY'**
**Thursday 15th March 2012**

**Organised by the United Kingdom Mathematics Trust and the Association Kangourou Sans Frontières**

*This competition is being taken by 5 million students in over 40 countries worldwide.*

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2. Time allowed: **1 hour**.
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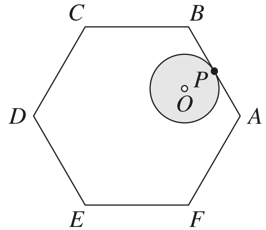
5. A circle with centre $O$ and radius 3 cm rolls without slipping around the inside of a regular hexagon $ABCDEF$. The circle starts with a point $P$ on its circumference in contact with the midpoint of the side $AB$, as shown.

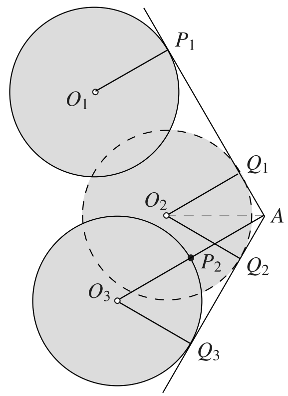
The circle then turns through one complete revolution anticlockwise, touching only sides $AB$ and $AF$ in the process. The circle ends in contact with the side $AF$, and $O$, $P$ and $A$ are in a straight line.




Find the length of $AB$.

*Solution*

Let $O_1$ and $P_1$ be the original positions of $O$ and $P$. When the disc reaches the corner at $A$, let $O_2$ be the position of the centre and let $Q_1$ and $Q_2$ be the points of contact with $AB$ and $AF$. Finally, when the disc has revolved exactly once, let $O_3$ be the centre, let $Q_3$ be the point of contact and let $P_2$ be the final position of $P$.




Now $\angle Q_1AQ_2 = 120°$ and $O_2Q_1 = 3$, so $AQ_1 = AQ_2 = \sqrt{3}$. Since the circle has completed one revolution without slipping $P_1Q_1 + Q_2Q_3 = 6\pi$.

Also $O_3A$ is perpendicular to $AP_1$, so $\angle O_3AQ_3 = 30°$. Hence $AQ_3 = 3\sqrt{3}$. Letting $P_1A = a$ and putting all of this together, we obtain

$$P_1Q_1 + Q_2Q_3 = \left(a - \sqrt{3}\right) + \left(3\sqrt{3} - \sqrt{3}\right) = 6\pi$$

and so $a = 6\pi - \sqrt{3}$. Therefore the length $2a$ of the side of the hexagon is $12\pi - 2\sqrt{3}$.

3. On Utopia Farm, Farmer Giles has a field in which the amount of grass always increases by the same amount each day. Six cows would take three whole days to eat all the grass in the field; three cows would take seven whole days to eat all the grass in the field.

Assuming that each of Farmer Giles' cows eats the same amount of grass per day, how long would one cow take to eat all the grass in the field?

*Solution*

Let $F$ be the amount of grass in the field initially, $C$ the amount a cow eats in one day and $A$ the increase of grass in one day. Then from the given information we have

$$F + 3A = 6 \times 3 \times C \qquad (1)$$

and $$F + 7A = 3 \times 7 \times C. \qquad (2)$$

Subtracting equation (1) from equation (2), we obtain $4A = 3C$ and hence $F = 21A$.

Now let $n$ be the number of days it would take one cow to eat all of the grass. Then we have $F + nA = nC$, and, using the values above, we obtain $n = 63$. Thus it would take 63 days for one cow to eat all the grass.

4. Find all real values of $x$ that satisfy the equation $(1+x)^4 - 2(1-x)^4 = (1-x^2)^2$.

*Solution*

Let $a = 1 + x$ and $b = 1 - x$. Then $ab = 1 - x^2$ and the given equation becomes $a^4 - 2b^4 = a^2b^2$, so that

$$a^4 - a^2b^2 - 2b^4 = 0,$$

that is,

$$\left(a^2 + b^2\right)\left(a^2 - 2b^2\right) = 0.$$

Hence either $a^2 + b^2 = 0$ or $a^2 - 2b^2 = 0$. In the first case $a = b = 0$, which is impossible; in the second case $a = \pm\sqrt{2}b$. In terms of $x$, this means that $1 + x = \pm\sqrt{2}(1-x)$, so that

$$\text{either} \quad x = \frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = 3 - 2\sqrt{2},$$

$$\text{or} \quad x = \frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} = 3 + 2\sqrt{2}.$$



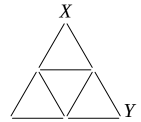
1. A watch is placed face up on a table so that its minute hand points north-east. How many minutes pass before the minute hand points north-west for the first time?

A 45 B 40 C 30 D 20 E 15

2. The Slovenian hydra has five heads. Every time a head is chopped off, five new heads grow. Six heads are chopped off one by one. How many heads will the hydra finally have?

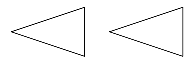
A 25 B 29 C 30 D 33 E 35

3. Each of the nine paths in a park is 100 m long. Ann wants to go from $X$ to $Y$ without going along any path more than once. What is the length of the longest route she can choose?



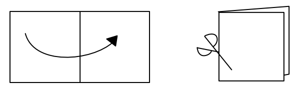

A 900 m B 800 m C 700 m D 600 m E 500 m

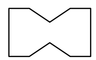
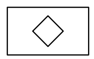
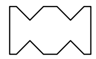
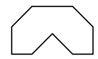
4. The diagram (which **is** drawn to scale) shows two triangles. In how many ways can you choose two vertices, one in each triangle, so that the straight line through the two vertices does not cross either triangle?



A 1 B 2 C 3 D 4 E more than 4

5. Werner folds a sheet of paper as shown in the diagram and makes two straight cuts with a pair of scissors. He then opens up the paper again. Which of the following shapes cannot be the result?



A  B  C  D  E 

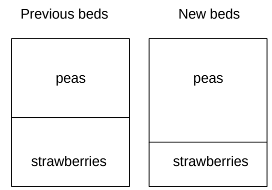
6. In each of the following expressions, the number 8 is to be replaced by a fixed positive number other than 8. In which expression do you get the same result, whatever positive number 8 is replaced by?

A $\frac{8+8}{8}+8$ B $8 \times \frac{8+8}{8}$ C $8+8-8+8$ D $(8+8-8)\times 8$ E $\frac{8+8-8}{8}$

7. Kanga forms two four-digit numbers using each of the digits 1, 2, 3, 4, 5, 6, 7 and 8 exactly once. Kanga wants the sum of the two numbers to be as small as possible. What is the value of this smallest possible sum?

A 2468 B 3333 C 3825 D 4734 E 6912

8. Mrs Gardner has beds for peas and strawberries in her rectangular garden. This year, by moving the boundary between them, she changed her rectangular pea bed to a square by lengthening one of its sides by 3 metres. As a result of this change, the area of the strawberry bed reduced by 15 m$^2$. What was the area of the pea bed before the change?




A 5 m$^2$ B 9 m$^2$ C 10 m$^2$ D 15 m$^2$ E 18 m$^2$

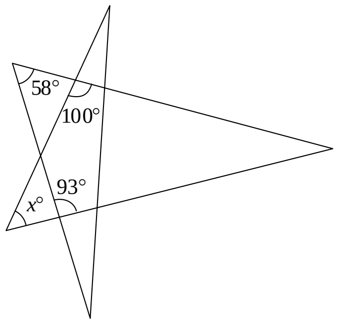
9. Barbara wants to complete the diagram below by inserting three numbers, one into each empty cell. She wants the sum of the first three numbers to be 100, the sum of the middle three numbers to be 200 and the sum of the last three numbers to be 300. What number should Barbara insert into the middle cell of the diagram?

| | | | | |
|---|---|---|---|---|
| 10 | | | | 130 |

A 50 B 60 C 70 D 75 E 100

10. In the figure, what is the value of $x$?

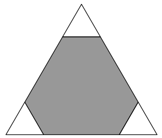
A 51 B 48 C 45 D 42 E 35




11. Four cards each have a number written on one side and a phrase written on the other. The four phrases are ‘divisible by 7’, ‘prime’, ‘odd’ and ‘greater than 100’ and the four numbers are 2, 5, 7 and 12. On each card, the number does not have the property given on the other side. What number is written on the same card as the phrase ‘greater than 100’?

A 2 B 5 C 7 D 12 E impossible to determine

12. Three small equilateral triangles of the same size are cut from the corners of a larger equilateral triangle with sides 6 cm as shown. The sum of the perimeters of the three small triangles is equal to the perimeter of the remaining hexagon. What is the side-length of one of the small triangles?



A 1 cm B 1.2 cm C 1.25 cm D 1.5 cm E 2 cm

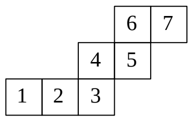
13. A piece of cheese was cut into a large number of pieces. During the course of the day, a number of mice came and stole some pieces, watched by the lazy cat Ginger. Ginger noticed that each mouse stole a different number of pieces, that each mouse stole fewer than 10 pieces and that no mouse stole exactly twice as many pieces as any other mouse. What is the largest number of mice that Ginger could have seen stealing cheese?

A 4 B 5 C 6 D 7 E 8

14. At the airport there is a moving walkway 500 metres long, which moves with a speed of 4 km/hour. Andrew and Bill step onto the walkway at the same time. Andrew walks with a speed of 6 km/hour on the walkway while Bill stands still. When Andrew comes to the end of the walkway, how far is he ahead of Bill?

A 100 m B 160 m C 200 m D 250 m E 300 m

15. A cube is being rolled on a plane so it turns around its edges. Its bottom face passes through the positions 1, 2, 3, 4, 5, 6 and 7 in that order, as shown. Which of these two positions were occupied by the same face of the cube?




A 1 and 7 B 1 and 6 C 1 and 5 D 2 and 7 E 2 and 6

16. Rick has five cubes. When he arranges them from smallest to largest, the difference between the heights of two neighbouring cubes is always 2 cm. The largest cube is as high as a tower built of the two smallest cubes. How high is a tower built of all five cubes?

A 50 cm B 44 cm C 22 cm D 14 cm E 6 cm



## Solutions to the Olympiad Maclaurin Paper

1. Both the digits $x$ and $y$ are nonzero. The five-digit integer ‘$xyxyx$’ is divisible by 3 and the seven-digit integer ‘$yxyxyxy$’ is divisible by 18.

Find all possible values of $x$ and $y$.

*Solution*

Since 3 is a factor of ‘$xyxyx$’ we know that 3 is a factor of the digit sum $3x + 2y$ and hence of $2y$. But 2 is coprime to 3, so 3 is a factor of $y$.

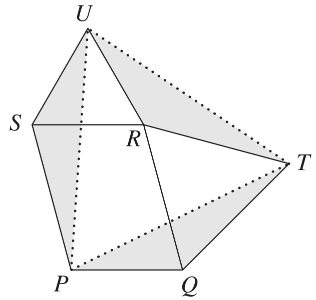
The integer ‘$yxyxyxy$’ is even, so $y$ is even. Hence we deduce that $y = 6$, the only even digit that has 3 as a factor.

Now 9 is a factor of ‘$6x6x6x6$’ and so it is also a factor of the digit sum $24 + 3x$, and hence of $6 + 3x$. Thus $x = 1$, 4 or 7, the only digits for which this is the case.

2. Points $T$ and $U$ lie outside parallelogram $PQRS$, and are such that triangles $RQT$ and $SRU$ are equilateral and lie wholly outside the parallelogram.

Prove that triangle $PTU$ is equilateral.

*Solution*




From the properties of a parallelogram and an equilateral triangle we have $PQ = SR = US = UR$ and $SP = RQ = QT = RT$. Also

$$\begin{aligned} \angle USP &= 60° + \angle RSP \\ &= 60° + \angle PQR \\ &= \angle PQT, \end{aligned}$$

and, from ‘angles at a point’,

$$\begin{aligned} \angle URT &= 360° - 60° - 60° - \angle QRS \\ &= 240° - (180° - \angle PQR) \\ &= 60° + \angle PQR \\ &= \angle PQT. \end{aligned}$$

Therefore triangles $USP$, $PQT$ and $URT$ are congruent (SAS) and it follows that $UP = PT = TU$.

6. Every cell of the following crossnumber is to contain a single digit. All the digits from 1 to 9 are used.

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | | |
| 5 | | |

Prove that there is exactly one solution to the crossnumber.

| Across | Down |
|---|---|
| 1 A multiple of 21. | 1 A multiple of 12. |
| 4 A multiple of 21. | 2 A multiple of 12. |
| 5 A multiple of 21. | 3 A multiple of 12. |

*Solution*

All multiples of 12 are even, so each digit of 5 ACROSS is 2, 4, 6 or 8. Now 5 ACROSS is a multiple of 21 so it is also a multiple of 3, and we know that a number is a multiple of 3 if, and only if, the sum of the digits is also a multiple of 3. But $2 + 4 + 6 + 8 = 20$, so that the only possibilities are not to use 2, or not to use 8. Hence the digits of 5 ACROSS are 4, 6 and 8, or 2, 4 and 6, in some order. We deduce that 5 ACROSS is 246, 264, 426, 462, 624, 642, 468, 486, 648, 684, 846 or 864. But 5 ACROSS is a multiple of 21 so it is also a multiple of 7. By checking them, we see that only 462 is divisible by 7. Thus 5 ACROSS is 462.

Now $12 = 4 \times 3$, so that all the DOWN answers are multiples of 4, and the number formed by the last two digits of a multiple of 4 is itself divisible by 4. Hence the last two digits of 1 down are 24, 44, 64 or 84. But 2, 4 and 6 have already been placed, therefore the last two digits of 1 DOWN are 84.

The first digit of 4 ACROSS is therefore 8. Now the multiples of 21 between 800 and 900 are 819, 840, 861 and 882. Once again, since 2, 4 and 6 have already been placed 4 ACROSS can only be 819. The last two digits in the DOWN columns are now 84, 16 and 92, all of which are divisible by 4.

Finally, 1 DOWN is a multiple of 3, so it is 384, 684 or 984. But 6 and 9 have already been placed so 1 down is 384. Likewise, 2 DOWN is 516 and 3 down is 792.

Therefore the completed crossnumber can only be

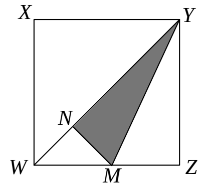
| | | |
|---|---|---|
| 3 | 5 | 7 |
| 8 | 1 | 9 |
| 4 | 6 | 2 |

and we may check that all the clues are satisfied. In particular, we observe that we have not used clue 1 ACROSS, so we need to check that 357 is divisible by 21.

Thus there is exactly one solution to the crossnumber.



17. In the diagram, $WXYZ$ is a square, $M$ is the midpoint of $WZ$ and $MN$ is perpendicular to $WY$. What is the ratio of the area of the shaded triangle $MNY$ to the area of the square?




A 1:6 B 1:5 C 7:36 D 3:16 E 7:40

18. The tango is danced by couples, each consisting of one man and one woman. At a dance evening, fewer than 50 people were present. At one moment, $\frac{3}{4}$ of the men were dancing with $\frac{4}{5}$ of the women. How many people were dancing at that moment?

A 20 B 24 C 30 D 32 E 40

19. David wants to arrange the twelve numbers from 1 to 12 in a circle so that any two neighbouring numbers differ by either 2 or 3. Which of the following pairs of numbers have to be neighbours?

A 5 and 8 B 3 and 5 C 4 and 6 D 7 and 9 E 6 and 8

20. Some three-digit integers have the following property: if you remove the first digit of the number, you get a perfect square; if instead you remove the last digit of the number, you also get a perfect square. What is the sum of all the three-digit integers with this curious property?

A 1013 B 1177 C 1465 D 1993 E 2016

21. A book contains 30 stories, each starting on a new page. The lengths of the stories are 1, 2, 3, ..., 30 pages in some order. The first story starts on the first page. What is the largest number of stories that can start on an odd-numbered page?

A 15 B 18 C 20 D 21 E 23

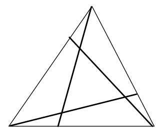
22. An equilateral triangle starts in a given position and is moved to new positions by a sequence of steps. At each step it is rotated clockwise about its centre; at the first step by 3°, at the second step by a further 9°; at the third by a further 27° and, in general, at the $n$ th step by a further $(3^n)°$. How many different positions, including the initial position, will the triangle occupy?

A 3 B 4 C 5 D 6 E 360

23. A long thin ribbon is folded in half lengthways, then in half again and then in half again. Finally, the folded ribbon is cut through at right angles to its length forming several strands. The lengths of two of the strands are 4 cm and 9 cm. Which of the following could **not** have been the length of the original ribbon?

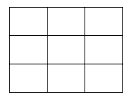
A 52 cm B 68 cm C 72 cm D 88 cm E all answers are possible

24. A large triangle is divided into four smaller triangles and three quadrilaterals by three straight line segments. The sum of the perimeters of the three quadrilaterals is 25 cm. The sum of the perimeters of the four triangles is 20 cm. The perimeter of the original triangle is 19 cm. What is the sum of the lengths of the three straight line segments?



A 11 cm B 12 cm C 13 cm D 15 cm E 16 cm

25. Each cell of the $3 \times 3$ grid shown has placed in it a positive number so that: in each row and each column, the product of the three numbers is equal to 1; and in each $2 \times 2$ square, the product of the four numbers is equal to 2. What number should be placed in the central cell?



A 16 B 8 C 4 D $\frac{1}{4}$ E $\frac{1}{8}$

**EUROPEAN ‘KANGAROO’ MATHEMATICAL CHALLENGE**
**‘PINK’**
**Thursday 15th March 2012**

**Organised by the United Kingdom Mathematics Trust and the Association Kangourou Sans Frontières**

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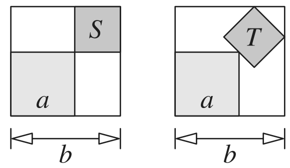
*Enquiries about the European Kangaroo should be sent to: Maths Challenges Office,*
*School of Mathematics Satellite, University of Leeds, Leeds, LS2 9JT.*
*(Tel. 0113 343 2339)*
*http://www.ukmt.org.uk*



5. Squares $S$ and $T$ are each placed outside a square of side $a$ and inside a square of side $b$, as shown. On the left, the *sides* of square $S$ are parallel to the sides of the other two squares; on the right, the *diagonals* of square $T$ are parallel to the sides of the other two squares.

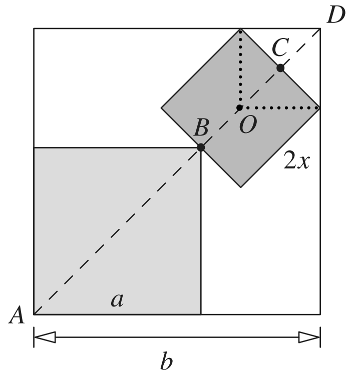



Find the ratio (area of $S$) : (area of $T$ ).

*Solution*

The area of square $S$ equals $(b - a)^2$.

Let the side length of square $T$ be $2x$. Consider the diagonal $AD$ of the square of side $b$, shown dashed in the figure.




This diagonal has three parts, the diagonal $AB$ of the square of side $a$, a line $BC$ crossing the square $T$, and the height $CD$ of a small triangle. That small triangle is congruent to one quarter of $T$, as indicated by the dotted lines.

We have $AD = b\sqrt{2}$ and $AB = a\sqrt{2}$ since these are diagonals of squares (using Pythagoras' Theorem). Also $BC = 2x$, and $CD = CO = x$, where $O$ is the centre of the square $T$.

But $AD = AB + BC + CD$, so we have

$$b\sqrt{2} = a\sqrt{2} + 2x + x$$

from which we deduce that $x = \frac{1}{3}\sqrt{2}(b - a)$. Hence the area of $T$ is

$$(2x)^2 = 4x^2 = 4 \times \tfrac{1}{9} \times 2(b - a)^2 = \tfrac{8}{9}(b - a)^2 .$$

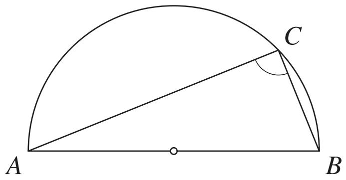
It follows that (area of $S$ ) : (area of $T$ ) = 9 : 8.

4. The eight points $A$, $B$, $C$, $D$, $E$, $F$, $G$ and $H$ are equally spaced on the perimeter of a circle, so that the arcs $AB$, $BC$, $CD$, $DE$, $EF$, $FG$, $GH$ and $HA$ are all equal.

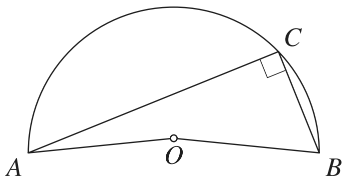
Joining any three of these points forms a triangle. How many of these triangles are right-angled?

*Solution*

That ‘the angle in a semicircle is 90°’ is well known: given a diameter $AB$ of a semicircle and another point $C$ on the circumference, as shown, then $\angle BCA = 90°$.




The converse result is also true, but may be less well known: given a right-angled triangle $ABC$ with hypotenuse $AB$, then $AB$ is a diameter of the circle through $A$, $B$ and $C$.




To see why this is true, join $A$ and $B$ to the centre $O$ of the circle. Then, using ‘the angle at the centre is twice the angle at the circumference’, we obtain $\angle BOA = 2\angle BCA = 2 \times 90° = 180°$. It follows that $AOB$ is a straight line, as required.

Now consider triangles formed by joining three of the eight points given in the question. The first result shows that the triangle will be right-angled when one side is a diameter, and the second result shows that this is the only way to obtain a right-angled triangle. So we may find the number of right-angled triangles by counting the number of ways of choosing two points at the ends of a diameter, and then choosing the third point.

Now there are 4 ways to choose a diameter connecting the given points: $AE$, $BF$, $CG$ and $DH$.

For a given choice of diameter, there are 6 different ways to choose the third point to form a triangle. For example, diameter $AE$ forms a right-angled triangle with each of the points $B$, $C$, $D$, $F$, $G$ and $H$.

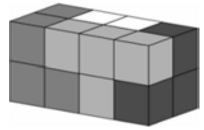
Hence altogether there are $4 \times 6 = 24$ ways to form a right-angled triangle.

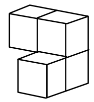
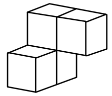
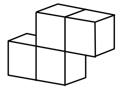
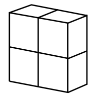
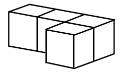


1. What is the value of $11.11 - 1.111$?

A 9.009 B 9.0909 C 9.99 D 9.999 E 10

2. A cuboid is made of four pieces as shown. Each piece consists of four cubes and is a single colour. What is the shape of the white piece?

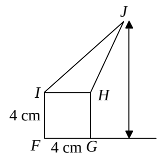


A  B  C  D  E 

3. The sum of the digits of a 7-digit number is six. What is the product of the digits?

A 0 B 1 C 5 D 6 E 7

4. The triangle $HIJ$ has the same area as the square $FGHI$, whose sides are of length 4 cm. What is the perpendicular distance, in cm, of the point $J$ from the line extended through $F$ and $G$?




A 8 B $4 + 2\sqrt{3}$ C 12 D $10\sqrt{2}$
E depends on the location of $J$

5. In four of the following expressions, the value of the expression is unchanged when each number 8 is replaced by any other positive number (always using the same number for every replacement). Which expression does *not* have this property?

A $(8 + 8 - 8) \div 8$ B $8 + (8 \div 8) - 8$ C $8 \div (8 + 8 + 8)$
D $8 \times (8 \div 8) \div 8$ E $8 - (8 \div 8) + 8$

6. The right-angled triangle $FGH$ has shortest sides of length 6 cm and 8 cm. The points $I$, $J$, $K$ are the midpoints of the sides $FG$, $GH$, $HF$ respectively. What is the length, in cm, of the perimeter of the triangle $IJK$ ?

A 10 B 12 C 15 D 20 E 24

7. When 144 is divided by the positive integer $n$, the remainder is 11. When 220 is divided by the positive integer $n$, the remainder is also 11. What is the value of $n$?

A 11 B 15 C 17 D 19 E 38

8. A quadrilateral has a side of length 1 cm and a side of length 4 cm. It has a diagonal of length 2 cm that dissects the quadrilateral into two isosceles triangles. What is the length, in cm, of the perimeter of the quadrilateral?

A 8 B 9 C 10 D 11 E 12

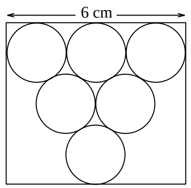
9. When Clement stands on a table and Dimitri stands on the floor, Clement appears to be 80 cm taller than Dimitri. When Dimitri stands on the same table and Clement stands on the floor, Dimitri appears to be one metre taller than Clement. How high is the table, in metres?

A 0.2 B 0.8 C 0.9 D 1 E 1.2

10. Maria and Meinke spun a coin thirty times. Whenever the coin showed heads, Maria gave two sweets to Meinke. When the coin showed tails, Meinke gave three sweets to Maria. After 30 spins, both Maria and Meinke had the same number of sweets as they started with. How many times were tails spun?

A 6　　B 12　　C 18　　D 24　　E 30

11. Six identical circles fit together tightly in a rectangle of width 6 cm as shown. What is the height, in cm, of the rectangle?

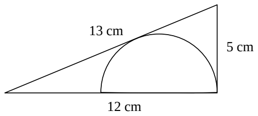



A 5　　B $2\sqrt{3}+2$　　C $3\sqrt{2}$　　D $3\sqrt{3}$　　E 6

12. In Clara's kitchen there is a clock on each of the four walls. Each clock is either slow or fast. The first clock is wrong by two minutes, the second clock by three minutes, the third by four minutes and the fourth by five minutes. What is the actual time when the four clocks show, in no particular order, six minutes to three, three minutes to three, two minutes past three and three minutes past three?

A 2:57　　B 2:58　　C 2:59　　D 3:00　　E 3:01

13. The right-angled triangle shown has sides of length 5 cm, 12 cm and 13 cm. What, in cm, is the radius of the inscribed semicircle whose diameter lies on the side of length 12 cm?




A 8/3　　B 10/3　　C 11/3　　D 4　　E 13/3

14. Numbers are to be placed into the table shown, one number in each cell, in such a way that each row has the same total, and each column has the same total. Some of the numbers are already given. What number is $x$?

| | | | |
|---|---|---|---|
| 2 | 4 | | 2 |
| | 3 | 3 | |
| 6 | | 1 | $x$ |

A 4　　B 5　　C 6　　D 7　　E 8

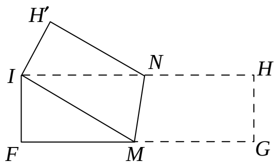
15. Three runners, Friedrich, Gottlieb and Hans had a race. Before the race, a commentator said, "Either Friedrich or Gottlieb will win." Another commentator said, "If Gottlieb comes second, then Hans will win." Another said, "If Gottlieb comes third, Friedrich will not win." And another said, "Either Gottlieb or Hans will be second." In the event, it turned out that all the commentators were correct. In what order did the runners finish?

A Friedrich, Gottlieb, Hans　　B Friedrich, Hans, Gottlieb
C Hans, Gottlieb, Friedrich　　D Gottlieb, Friedrich, Hans
E Gottlieb, Hans, Friedrich

16. What is the last non-zero digit when $2^{57} \times 3^4 \times 5^{53}$ is evaluated?

A 1　　B 2　　C 4　　D 6　　E 8

17. A rectangular piece of paper $FGHI$ with sides of length 4 cm and 16 cm is folded along the line $MN$ so that the vertex $G$ coincides with the vertex $I$ as shown. The outline of the paper now makes a pentagon $FMNH'I$. What is the area, in cm$^2$, of the pentagon $FMNH'I$?




A 51　　B 50　　C 49　　D 48　　E 47



3. On Monday, the cost of 3 bananas was the same as the total cost of a lemon and an orange.

On Tuesday, the cost of each fruit was reduced by the same amount, resulting in the cost of 2 oranges being the same as the total cost of 3 bananas and a lemon.

On Wednesday, the cost of a lemon halved to 5 p.

What was the cost of an orange on Monday?

*Solution*

Let $x$ p, $y$ p and $z$ p be the costs on Monday of a banana, lemon and orange respectively. Let the common amount by which the cost of each fruit was reduced on Tuesday be $r$ p.

From the given information, we have

$$3x = y + z, \tag{1}$$

$$2(z - r) = 3(x - r) + y - r, \tag{2}$$

and $$\tfrac{1}{2}(y - r) = 5. \tag{3}$$

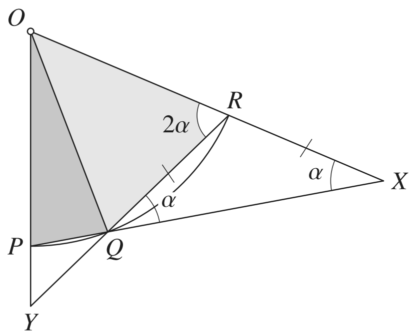
Expanding equation (2), we get $2z - 2r = 3x - 3r + y - r$, so that $2z = 3x - 2r + y$. Substituting for $3x$ from equation (1), we obtain $2z = y + z - 2r + y$, and hence $z = 2y - 2r = 2(y - r)$.

But $y - r = 10$ from equation (3), thus $z = 2 \times 10 = 20$.

Hence the cost of an orange on Monday was 20 p.

*Method* 2

Since $PQR$ is the arc of a circle, centre $O$, we have $OP = OQ = OR$ (radii).




In particular, $OQ = OR$, so that triangle $OQR$ is isosceles and $\angle RQO = \angle QRO = 2\alpha$.
The sum of the angles on a straight line equals 180°, hence

$$\begin{aligned}\angle OQP &= 180° - \angle RQX - \angle RQO \\ &= 180° - \alpha - 2\alpha \\ &= 180° - 3\alpha.\end{aligned}$$

Now $OP = OQ$, so that triangle $OQP$ is also isosceles and
$\angle OPQ = \angle OQP = 180° - 3\alpha$.
The sum of the angles in a triangle equals 180°, hence, in triangle $OPX$,

$$\begin{aligned}\angle POX &= 180° - \angle OPX - \angle OXP \\ &= 180° - \angle OPQ - \angle RXQ \\ &= 180° - (180° - 3\alpha) - \alpha \\ &= 2\alpha.\end{aligned}$$

Thus $\angle YOR = \angle POX = 2\alpha = \angle QRO = \angle YRO$, so that $OY = RY$ from 'sides opposite equal angles'.



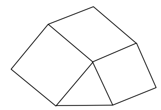
18. Erica saw an eastbound train to Brussels passing. It took 8 seconds to pass her. A westbound train to Lille took 12 seconds to pass her. They took 9 seconds to pass each other. Assuming both trains maintained a constant speed, which of the following statements is true?

A the Brussels train is twice as long as the Lille train B the trains are of the same length
C the Lille train is 50% longer than the Brussels train D the Lille train is twice as long as the Brussels train E it is impossible to say if the statements A to D are true

19. Brigitte wrote down a list of all 3-digit numbers. For each of the numbers on her list she found the product of the digits. She then added up all of these products. Which of the following is equal to this total?

A 45 B $45^2$ C $45^3$ D $2^{45}$ E $3^{45}$

20. The diagram shows a square with sides of length 4 mm, a square with sides of length 5 mm, a triangle with area 8 mm$^2$, and a parallelogram. What is the area, in mm$^2$, of the parallelogram?



A 15 B 16 C 17 D 18 E 19

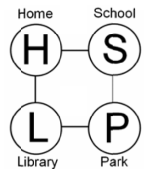
21. Anya has found positive integers $k$ and $m$ such that $m^m \times (m^k - k) = 2012$. What is the value of $k$?

A 2 B 3 C 8 D 9 E 11

22. Pedro writes down a list of six different positive integers, the largest of which is $N$. There is exactly one pair of these numbers for which the smaller number does not divide the larger. What is the smallest possible value of $N$?

A 18 B 20 C 24 D 36 E 45

23. Carlos creates a game. The diagram shows the board for the game. At the start, the kangaroo is at the school (S). According to the rules of the game, from any position except home (H), the kangaroo can jump to either of the two neighbouring positions. When the kangaroo lands on H the game is over. In how many ways can the kangaroo move from S to H in exactly 13 jumps?




A 12 B 32 C 64 D 144 E 1024

24. Lali and Gregor play a game with five coins, each with Heads on one side and Tails on the other. The coins are placed on a table, with Heads showing. In each round of the game, Lali turns over a coin, and then Gregor turns over a different coin. They play a total of ten rounds. Which of the following statements is then true?

A It is impossible for all the coins to show Heads
B It is impossible for all the coins to show Tails
C It is definite that all the coins show Heads
D It is definite that all the coins show Tails
E None of the statements A to D is true

25. A regular octagon has vertices $A, B, C, D, E, F, G, H$. One of the vertices $C, D, E, F, G, H$ is chosen at random, and the line segment connecting it to $A$ is drawn. Then one of the vertices $C, D, E, F, G, H$ is chosen at random and the line segment connecting it to $B$ is drawn. What is the probability that the octagon is cut into exactly three regions by these two line segments?

A $\frac{1}{6}$ B $\frac{5}{18}$ C $\frac{1}{4}$ D $\frac{1}{3}$ E $\frac{4}{9}$

**The United Kingdom Mathematics Trust**

# Intermediate Mathematical Olympiad and Kangaroo (IMOK)

## Olympiad Cayley/Hamilton/Maclaurin Papers

Thursday 15th March 2012

**READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING**

1. Time allowed: 2 hours.
2. **The use of calculators, protractors and squared paper is forbidden.** Rulers and compasses may be used.
3. Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Cover Sheet on top.
4. Start each question on a fresh A4 sheet.

   You may wish to work in rough first, then set out your final solution with clear explanations and proofs.

   ***Do not hand in rough work.***
5. Answers must be FULLY SIMPLIFIED, and EXACT. They may contain symbols such as $\pi$, fractions, or square roots, if appropriate, but NOT decimal approximations.
6. Give full written solutions, including mathematical reasons as to why your method is correct.

   Just stating an answer, even a correct one, will earn you very few marks; also, incomplete or poorly presented solutions will not receive full marks.
7. **These problems are meant to be challenging!** The earlier questions tend to be easier; the last two questions are the most demanding. Do not hurry, but spend time working carefully on one question before attempting another. Try to finish whole questions even if you cannot do many: you will have done well if you hand in full solutions to two or more questions.

**DO NOT OPEN THE PAPER UNTIL INSTRUCTED BY THE INVIGILATOR TO DO SO!**

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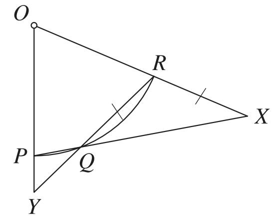
*Enquiries should be sent to: Maths Challenges Office,*
*School of Maths Satellite, University of Leeds, Leeds, LS2 9JT.*
*(Tel. 0113 343 2339)*
*http://www.ukmt.org.uk*



2. The diagram shows an arc $PQR$ of a circle, centre $O$. The lines $PQ$ and $OR$ meet at $X$, with $QR = RX$, and the lines $OP$ and $RQ$ meet at Y.

   Prove that $OY = RY$.




*Solution*

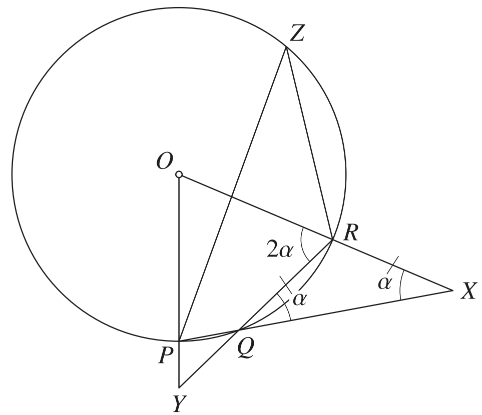
Let $\angle RXQ = \alpha$. Then since $QR = RX$ we have $\angle RQX = \angle RXQ = \alpha$.

Using 'an external angle of a triangle equals the sum of the two interior opposite angles' in triangle $RXQ$, we obtain $\angle QRO = \angle RXQ + \angle RQX = \alpha + \alpha = 2\alpha$.

We now proceed in one of two ways: the first method uses circle theorems; the second method uses nothing more than facts about triangles.

*Method* 1

Let $Z$ be any point on the circle which is not on arc $PQR$.




Then $\angle RQX$ is an exterior angle of the cyclic quadrilateral $PQRZ$ and therefore $\angle RQX = \angle PZR$. Hence $\angle PZR = \alpha$.

Since 'the angle at the centre is twice the angle at the circumference' we deduce that $\angle POR = 2\alpha$.

We therefore have $\angle YOR = \angle POR = 2\alpha = \angle QRO = \angle YRO$, so that $OY = RY$ from 'sides opposite equal angles'.

## Solutions to the Olympiad Hamilton Paper

1. The digits *p*, *q*, *r*, *s* and *t* are all different.
What is the smallest five-digit integer '*pqrst*' that is divisible by 1, 2, 3, 4 and 5?

*Solution*

Note that all five-digit integers are divisible by 1.

Next, notice that if a number is divisible by 2 and 5, then it is divisible by 10 and hence has last digit *t* equal to 0. Since all the digits are different, that means all the other digits are non-zero.

Among five-digit numbers, those that begin with 1 are smaller than all those which do not. So if we find a number of the required form with first digit 1, then it will be smaller than numbers with larger first digits.

Similarly, those with first two digits 12 are smaller than all other numbers with distinct non-zero digits. And, in fact, those with first three digits 123 are smaller than all others. Hence if we find such a number with the required properties, it will be smaller than all others.

So let us try to find a number of the form '123*s*0'.

A number is a multiple of four only if its last two digits form a multiple of four. So we need consider only the case where s is even.

Similarly, a number is a multiple of three if and only if the sum of its digits is a multiple of three. Since $1 + 2 + 3 + s + 0 = 6 + s$, we only need consider the case where *s* is a multiple of three.

Thus 12360 is the only number of the form '123*s*0' which is divisible by 1, 2, 3, 4 and 5, and as we have explained along the way, it is the smallest number with the required divisibility properties.



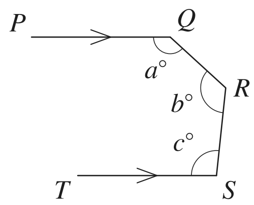
- *Do not hurry, but spend time working carefully on one question before attempting another.*
- *Try to finish whole questions even if you cannot do many.*
- *You will have done well if you hand in full solutions to two or more questions.*
- *Answers must be FULLY SIMPLIFIED, and EXACT. They may contain symbols such as $\pi$, fractions, or square roots, if appropriate, but NOT decimal approximations.*
- *Give full written solutions, including mathematical reasons as to why your method is correct.*
- *Just stating an answer, even a correct one, will earn you very few marks.*
- *Incomplete or poorly presented solutions will not receive full marks.*
- ***Do not hand in rough work.***

## Olympiad Cayley Paper

**All candidates must be in *School Year 9 or below* (England and Wales), *S2 or below* (Scotland), or *School Year 10 or below* (Northern Ireland).**

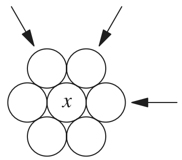
1. The digits $p$, $q$, $r$, $s$ and $t$ are all different.
   What is the smallest five-digit integer '*pqrst*' that is divisible by 1, 2, 3, 4 and 5?

2. In the diagram, $PQ$ and $TS$ are parallel.
   Prove that $a + b + c = 360$.

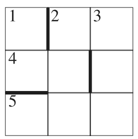



3. Three loaves of bread, five cartons of milk and four jars of jam cost £10.10.
   Five loaves of bread, nine cartons of milk and seven jars of jam cost £18.20.
   How much does it cost to buy one loaf of bread, one carton of milk and one jar of jam?

4. The diagram shows seven circles. Each of the three arrows indicates a 'line of three circles'.
   The digits from 1 to 7 inclusive are to be placed in the circles, one per circle, so that the sum of the digits in each of the three indicated 'lines of three circles' is the same.
   Find all possible values of $x$.




5. Every cell of the following crossnumber is to contain a single digit. No clue has an answer starting with zero.
   Prove that there is exactly one solution to the crossnumber.

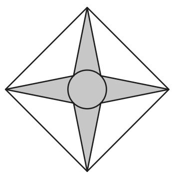



*Across*
- 2 Sum of the digits of 2 Down.
- 4 Prime.
- 5 1 Down + 2 Across + 3 Down.

*Down*
- 1 Product of two primes.
- 2 Multiple of 99.
- 3 Square of 4 Across.

6. The diagram shows a symmetrical four-pointed star. Four vertices of the star form a square and the other four vertices lie on a circle. The square has sides of length $2a$ cm. The shaded area is one third of the area of the square.

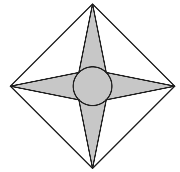
   What is the radius of the circle?





6. The diagram shows a symmetrical four-pointed star. Four vertices of the star form a square and the other four vertices lie on a circle. The square has sides of length $2a$ cm. The shaded area is one third of the area of the square.

   What is the radius of the circle?

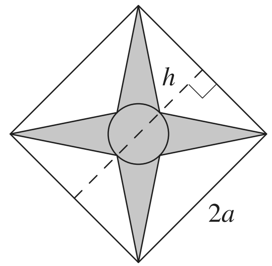


*Solution*

*Method* 1

Since the shaded area is one third of the area of the square, the area not shaded is two thirds. So each white triangle has an area of

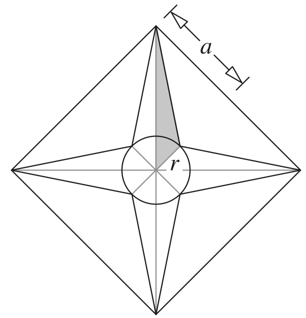
$$\tfrac{1}{4} \times \tfrac{2}{3} \times 4a^2 = \tfrac{2}{3}a^2.$$




However, the base of a white triangle is $2a$, so the height $h$ cm is given by $\tfrac{1}{2} \times 2a \times h = \tfrac{2}{3}a^2$. Hence $h = \tfrac{2}{3}a$.
Now the dashed line in the figure comprises two of these heights and a diameter of the circle. But the dashed line has a length of $2a$, the length of a side of the square. Thus the radius of the circle is $\tfrac{1}{3}a$.

*Method* 2

We first cut the star into eight congruent triangles, as shown.



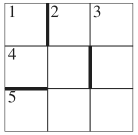

Each triangle may be considered to have base $r$, the radius of the circle, and height $a$, as indicated for the shaded triangle.
Using the formula for the area of a triangle, and the fact that these eight triangles together have area one third of the area of the whole square, we get

$$8 \times \tfrac{1}{2} \times r \times a = \tfrac{1}{3} \times 4a^2,$$

which we solve to obtain $r = \tfrac{1}{3}a$.

5. Every cell of the following crossnumber is to contain a single digit. No clue has an answer starting with zero.

Prove that there is exactly one solution to the crossnumber.



*Across*

2 Sum of the digits of 2 Down.

4 Prime.

5 1 Down + 2 Across + 3 Down.

*Down*

1 Product of two primes.

2 Multiple of 99.

3 Square of 4 Across.

*Solution*

Firstly, the answer to 2 DOWN has to be one of 198, 297, 396, 495, 594, 693, 792, 891 or 990. In all of these cases, the sum of the digits is the same: the answer to 2 ACROSS is thus 18.

Knowing the first digit is 1 immediately allows us to narrow down 2 DOWN to 198. Also, 3 DOWN begins with an 8 and is the square of 4 ACROSS, an integer which ends in 9. The number $19^2 = 361$ is too small, while $39^2$ and subsequent squares have four or more digits. Hence 4 ACROSS is 29 and 3 DOWN is 841.

We now use the clue for 5 ACROSS: we get

$$`x2' + 18 + 841 = `y81',$$

where $x$ is the missing top left-hand digit and $y$ is the missing bottom left-hand digit. Clearly $y$ is 8, since even if $x$ were 9 it would not be big enough to give a total of 981.

We can then subtract to find the answer to 1 DOWN: it is given by

$$881 - 841 - 18 = 22.$$

We have uniquely identified values for all nine digits, using the clues.

| | | |
|---|---|---|
| 2 | 1 | 8 |
| 2 | 9 | 4 |
| 8 | 8 | 1 |

In checking, we observe that we have not used the clue for 1 DOWN; however $22 = 2 \times 11$, so it does indeed work.



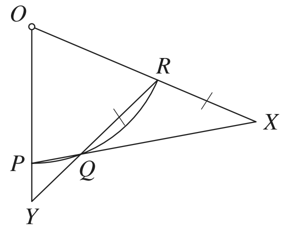
# Olympiad Hamilton Paper

**All candidates must be in *School Year 10* (England and Wales), *S3* (Scotland), or *School Year 11* (Northern Ireland).**

1. The digits $p$, $q$, $r$, $s$ and $t$ are all different.
What is the smallest five-digit integer '$pqrst$' that is divisible by 1, 2, 3, 4 and 5?

2. The diagram shows an arc $PQR$ of a circle, centre $O$. The lines $PQ$ and $OR$ meet at $X$, with $QR = RX$, and the lines $OP$ and $RQ$ meet at Y.

Prove that $OY = RY$.

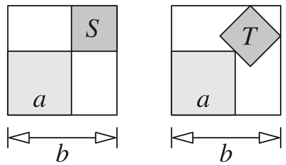



3. On Monday, the cost of 3 bananas was the same as the total cost of a lemon and an orange.
On Tuesday, the cost of each fruit was reduced by the same amount, resulting in the cost of 2 oranges being the same as the total cost of 3 bananas and a lemon.
On Wednesday, the cost of a lemon halved to 5p.
What was the cost of an orange on Monday?

4. The eight points $A$, $B$, $C$, $D$, $E$, $F$, $G$ and $H$ are equally spaced on the perimeter of a circle, so that the arcs $AB$, $BC$, $CD$, $DE$, $EF$, $FG$, $GH$ and $HA$ are all equal.

Joining any three of these points forms a triangle. How many of these triangles are right-angled?

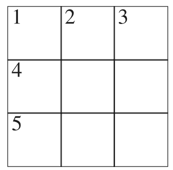
5. Squares $S$ and $T$ are each placed outside a square of side $a$ and inside a square of side $b$, as shown. On the left, the *sides* of square $S$ are parallel to the sides of the other two squares; on the right, the *diagonals* of square $T$ are parallel to the sides of the other two squares.




Find the ratio (area of $S$) : (area of $T$ ).

6. Every cell of the following crossnumber is to contain a single digit. All the digits from 1 to 9 are used.

Prove that there is exactly one solution to the crossnumber.



Across

1 A multiple of 21.

4 A multiple of 21.

5 A multiple of 21.

Down

1 A multiple of 12.

2 A multiple of 12.

3 A multiple of 12.

# Olympiad Maclaurin Paper

**All candidates must be in *School Year 11* (England and Wales), *S4* (Scotland), or *School Year 12* (Northern Ireland).**

1. Both the digits $x$ and $y$ are nonzero. The five-digit integer '$xyxyx$' is divisible by 3 and the seven-digit integer '$yxyxyxy$' is divisible by 18.

   Find all possible values of $x$ and $y$.

2. Points $T$ and $U$ lie outside parallelogram $PQRS$, and are such that triangles $RQT$ and $SRU$ are equilateral and lie wholly outside the parallelogram.

   Prove that triangle $PTU$ is equilateral.

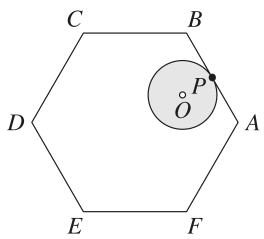
3. On Utopia Farm, Farmer Giles has a field in which the amount of grass always increases by the same amount each day. Six cows would take three whole days to eat all the grass in the field; three cows would take seven whole days to eat all the grass in the field.

   Assuming that each of Farmer Giles' cows eats the same amount of grass per day, how long would one cow take to eat all the grass in the field?

4. Find all real values of $x$ that satisfy the equation $(1+x)^4 - 2(1-x)^4 = (1-x^2)^2$.

5. A circle with centre $O$ and radius 3 cm rolls without slipping around the inside of a regular hexagon $ABCDEF$. The circle starts with a point $P$ on its circumference in contact with the midpoint of the side $AB$, as shown.

   The circle then turns through one complete revolution anticlockwise, touching only sides $AB$ and $AF$ in the process. The circle ends in contact with the side $AF$, and $O$, $P$ and $A$ are in a straight line.

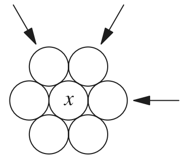
   Find the length of $AB$.

6. Three different positive integers have the property that each of them divides the sum of the other two.

   Find all such sets of three numbers.



4. The diagram shows seven circles. Each of the three arrows indicates a 'line of three circles'.

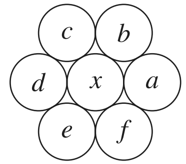
   
   

   The digits from 1 to 7 inclusive are to be placed in the circles, one per circle, so that the sum of the digits in each of the three indicated 'lines of three circles' is the same.

   Find all possible values of $x$.

   *Solution*

   Let us write $a$, $b$, $c$, $d$, $e$ and $f$ for the six values in the outer circles, as shown, and write $s$ for the common sum of the three lines.

   The given conditions then say that

   $$a + x + d = s,$$

   $$b + x + e = s,$$

   and $$c + x + f = s.$$

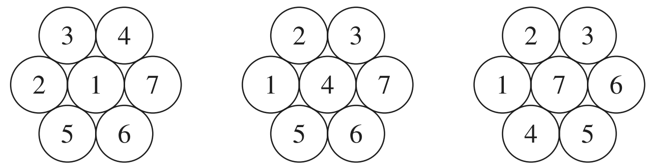
   If we add these equations together, we get

   $$a + b + c + d + e + f + 3x = 3s.$$

   However, the sum $a + b + c + d + e + f + x$ is equal to 28, since the numbers are 1, 2, 3, 4, 5, 6 and 7 in some order. So, rewriting, we get

   $$2x + 28 = 3s,$$

   which says that $2x + 28$ is a multiple of 3. We can quickly check that this happens only when $x = 1$, 4 or 7. Hence all other values of $x$ are impossible. Lastly, we demonstrate with three examples that these three values of $x$ are all possible:

3. Three loaves of bread, five cartons of milk and four jars of jam cost £10.10.
Five loaves of bread, nine cartons of milk and seven jars of jam cost £18.20.
How much does it cost to buy one loaf of bread, one carton of milk and one jar of jam?

*Solution*

We write $B$ for the price of a loaf of bread in pence, $M$ for the price of a carton of milk in pence, and $J$ for the price of a jar of jam in pence. Then we can interpret the given information as saying

$$3B + 5M + 4J = 1010$$

and $$5B + 9M + 7J = 1820.$$

If we double the first equation we get

$$6B + 10M + 8J = 2020,$$

and if we then subtract the second equation from this we get

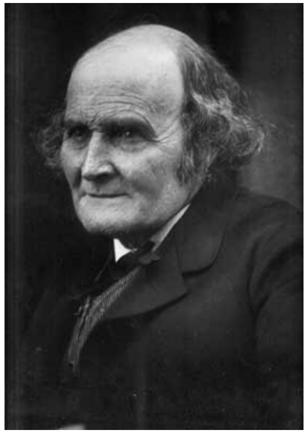
$$B + M + J = 200,$$

or, in words, the total cost of a loaf of bread, a carton of milk and a jar of jam is £2.



# Arthur Cayley

16th August 1821 − 26th January 1895



From a young age, Arthur Cayley showed great numerical flair and his mathematics teacher advised Cayley to continue his studies in this area, going against his father's wishes that he enter the family business.

Cayley did continue to study mathematics, at Trinity College, Cambridge, graduating in 1842. For four years he then taught at Cambridge and had 28 papers published in the Cambridge Mathematical Journal during this period. He then spent 14 years as a lawyer, choosing law as a profession in order to make money so he could pursue mathematics, publishing about 250 mathematical papers during this time.

In 1863, Cayley was appointed Sadleirian professor of Pure Mathematics at Cambridge. He published over 900 papers and notes during his lifetime, covering many aspects of modern mathematics. He is remembered mainly for his work in developing the algebra of matrices and for his studies in geometry.

Around 1849 Cayley wrote a paper linking his ideas on permutations with Cauchy's. Five years later, he wrote two papers about abstract groups. At that time the only known groups were permutation groups and even these formed a radically new area, yet Cayley defined an abstract group and gave a table to display the group multiplication. He gave the ‘Cayley tables’ of some special permutation groups but much more significantly for the introduction of the abstract group concept, he realised that invertible matrices and quaternions form groups (with the usual operation of multiplication).

Cayley also developed the theory of algebraic invariance, and his development of $n$-dimensional geometry has been applied in physics to the study of the space-time continuum. His work on matrices served as a foundation for quantum mechanics, which was developed by Werner Heisenberg in 1925. Cayley also suggested that euclidean and non-euclidean geometry are special types of geometry. He united projective geometry and metrical geometry, which is dependent on sizes of angles and lengths of lines.

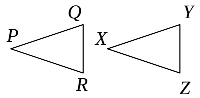
## Solutions to the European Kangaroo Grey Paper

**1. A** North-East is on a bearing of 045° and North-West is on a bearing of 315°. The watch hand has to turn (clockwise) $315° - 045° = 270°$ so this will take 270/360 of an hour i.e. ¾ of an hour which is equivalent to 45 minutes.

**2. B** Each time a head is chopped off, five extra heads grow so the net result of each chop is an increase of four heads. After 6 chops, the total number of heads will be $5 + 6 \times 4 = 29$.

**3. C** Altogether there are nine paths, making 900 m of path in total. The route is to start at X and not repeat any path. This means that only one of the paths from X can be used in the route. [Otherwise the other path would be used to bring Ann back to X and there would be no path remaining for her to leave X again.] Similarly, Ann can only use one path into Y. Therefore, a maximum of seven paths could be used. It is easy to see that there are several routes which use seven paths (leaving out the two right-hand paths, for example). Hence the maximum length of the route is $7 \times 100 \text{ m} = 700 \text{ m}$.

**4. D** Consider each vertex of the left-hand triangle in turn.




From vertex $P$, no line can be drawn.

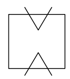
From vertex $Q$, a line can be drawn to vertex $X$ and vertex $Y$.

From vertex $R$, a line can be drawn to vertex $X$ and vertex $Z$.

Therefore, the two vertices can be chosen in only four ways.

**5. D** The shapes given in options A, B, C and E can be obtained by cutting the paper as shown.

The only one unobtainable in two cuts is D, which requires four cuts as shown.



**6. E** Replacing each occurrence of the number 8 by $y$ and simplifying, the expressions reduce to $2 + y$, $2y$, $2y$, $y^2$ and 1. The only one independent of $y$ is the last, which will always have the value 1 whatever positive number replaces 8 in the original expression.

**7. C** To get the minimum sum, the smaller digits must come before the larger digits in the two numbers.

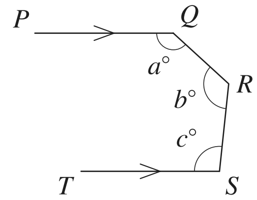
There are a number of different arrangements possible for the two numbers but all must satisfy the following restrictions: the 1000s digits must be 1 and 2, the 100s digits must be 3 and 4, the 10s digits must be 5 and 6 and the units digits must be 7 and 8.

One possible arrangement giving the minimum sum is $1357 + 2468$ which gives a total of 3825.

**8. C** If the length of the pea bed is increased by 3 m, the length of the strawberry bed is decreased by 3 m. As the area of the strawberry bed is reduced by 15 $\text{m}^2$, this means that the width of the strawberry bed (and hence of the pea bed) was $(15 \div 3)$ m = 5 m. The pea bed is now a square so must have area $5 \times 5 \text{m}^2 = 25 \text{ m}^2$. As its area has increased by 15 $\text{m}^2$ to reach this value, its original area must have been 10 $\text{m}^2$.



2. In the diagram, $PQ$ and $TS$ are parallel.
Prove that $a + b + c = 360$.

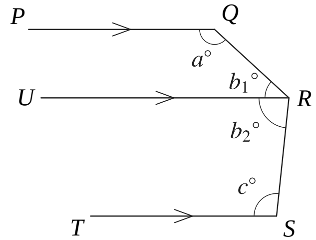



*Solution*

We describe just two of the many different methods that are possible.

*Method* 1

Draw a line $UR$ parallel to $PQ$ and $TS$ passing through $R$. This line divides the angle $b°$ into two parts; suppose these parts have sizes $b_1°$ and $b_2°$, as shown.




Then angles $a°$ and $b_1°$ are supplementary (because $PQ$ and $UR$ are parallel), as are angles $b_2°$ and $c°$. Therefore

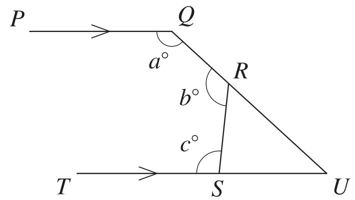
$$a + b_1 = 180$$

$$\text{and} \quad b_2 + c = 180.$$

By adding these two equations, we obtain the desired result since $b_1 + b_2 = b$.

*Method* 2

Extend $QR$ and $TS$ until they meet at a new point $U$.




Then angles $\angle PQR$ and $\angle SUR$ are supplementary (because $PQ$ and $TU$ are parallel), that is, $a° + \angle SUR = 180°$.

However, we have

$$\angle SUR + \angle URS + \angle RSU = 180°$$

by properties of angles in a triangle. This means that $a° = \angle URS + \angle RSU$.

We also have $b° + \angle URS = 180° = c° + \angle RSU$ since they are angles on a straight line.

Putting all this together we get

$$a° + b° + c° = \angle URS + \angle RSU + (180° - \angle URS) + (180° - \angle RSU) = 360°,$$

as needed.

# Solutions to the Olympiad Cayley Paper

1. The digits $p$, $q$, $r$, $s$ and $t$ are all different.
What is the smallest five-digit integer ‘$pqrst$’ that is divisible by 1, 2, 3, 4 and 5?

*Solution*

Note that all five-digit integers are divisible by 1.

Next, notice that if a number is divisible by 2 and 5, then it is divisible by 10 and hence has last digit $t$ equal to 0. Since all the digits are different, that means all the other digits are non-zero.

Among five-digit numbers, those that begin with 1 are smaller than all those which do not. So if we find a number of the required form with first digit 1, then it will be smaller than numbers with larger first digits.

Similarly, those with first two digits 12 are smaller than all other numbers with distinct non-zero digits. And, in fact, those with first three digits 123 are smaller than all others. Hence if we find such a number with the required properties, it will be smaller than all others.

So let us try to find a number of the form ‘123$s$0’.

A number is a multiple of four only if its last two digits form a multiple of four. So we need consider only the case where $s$ is even.

Similarly, a number is a multiple of three if and only if the sum of its digits is a multiple of three. Since $1 + 2 + 3 + s + 0 = 6 + s$, we only need consider the case where $s$ is a multiple of three.

Thus 12360 is the only number of the form ‘123$s$0’ which is divisible by 1, 2, 3, 4 and 5, and as we have explained along the way, it is the smallest number with the required divisibility properties.



**9. B**

| | | | | |
|---|---|---|---|---|
| 10 | $X$ | $Y$ | $Z$ | 130 |

If we label the values in the cells as shown, the question tells us that $10 + X + Y = 100$, $X + Y + Z = 200$ and $Y + Z + 130 = 300$. The first two equations give $Z = 110$ and substituting this into the third equation then gives $Y = 60$.

**10. A** Let the angle on the far right of the shape be $y°$. Using angles in a triangle, we have $58 + 93 + y = 180$, so $y = 29$. Using angles in a triangle again, we have $y + 100 + x = 180$, so $x = 51$.

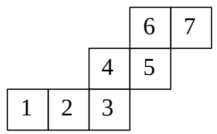
**11. C** The only number that is not a prime number is 12 and so 12 must go on the reverse of the card marked ‘prime’. This leaves 2 as the only remaining number that is not odd and so 2 must go on the reverse of the card marked ‘odd’. Then 5 is the only remaining number not divisible by 7 and so 5 must go on the reverse of the card marked ‘divisible by 7’. This leaves 7 to go on the reverse of the card marked ‘greater than 100’.

**12. D** If we let the length of the side of one of the removed triangles be $x$ cm, the perimeter of the remaining hexagon will be $3x + 3(6 - 2x)$ cm. Hence we have $3(3x) = 3x + 3(6 - 2x)$ which has solution $x = 18/12 = 1.5$.

**13. C** The possible numbers of pieces cannot include both 1 and 2, nor 3 and 6, nor 4 and 8. So certainly no more than $9 - 3 = 6$ mice are involved. However it is possible that (for example) 6 mice stole 1, 3, 4, 5, 7 and 9 pieces. Hence the largest possible number of mice seen is 6.

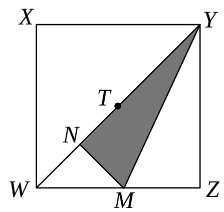
**14. E** Andrew's speed relative to an observer standing at the side of the walkway is $(6 + 4)$ km/h = 10 km/h. Let the distance Bill has covered when Andrew leaves the walkway be $x$ km. Since the ratio of speeds is equal to the ratio of distances covered in a fixed time, we have $10 : 4 = 0.5 : x$ which has solution $x = 0.2$. Therefore Bill will be $(0.5 - 0.2)$ km = 300 m behind.

**15. B** Imagine the grid is sticky so that when the cube rolls over it, each cell of the grid fastens to the face of the cube touching it. The result would be equivalent to taking the arrangement of cells as shown, cutting it out and folding it into a cube. The latter is possible (for example) with 5 on the bottom, 6 at the back, 7 on the right, 4 on the left, 3 at the front, 2 on the top and 1 folding over the 6 at the back. Hence 1 and 6 are occupied by the same face of the cube.




**16. A** Let the height of the smallest cube be $x$ cm. Hence the other heights are $(x + 2)$ cm, $(x + 4)$ cm, $(x + 6)$ cm and $(x + 8)$ cm. The information in the question tells us that $x + 8 = x + x + 2$ which has the solution $x = 6$. Hence the total height of the tower of all five cubes is $6 + 8 + 10 + 12 + 14 = 50$ cm.

**17. D** Introduce point $T$, the mid-point of $WY$. $MT$ is parallel to $ZY$ and half the length. The area of triangle $WMT$ is $\frac{1}{4}$ of the area of triangle $WZY = \frac{1}{8}$ of the area of the square. Also, the area of triangle $WMN$ is $\frac{1}{2}$ of the area of triangle $WMT = \frac{1}{16}$ of the area of the square. The area of triangle $WMY$ is $\frac{1}{4}$ of the area of the square so the area of triangle $NMY$ is $\left(\frac{1}{4} - \frac{1}{16}\right) = \frac{3}{16}$ of the area of the square. Hence the ratio of the area of triangle $MNY$ to the area of the square is 3 : 16.



*Alternative solution*: Suppose that the square has side length $s$ and hence area $s^2$. The triangle $WMY$ has a base of length $\frac{1}{2}s$ and height $s$, and hence area $\frac{1}{2}\left(\frac{1}{2}s \times s\right) = \frac{1}{4}s^2$. Triangle $WNM$ is a right-angled isosceles triangle with hypotenuse of length $\frac{1}{2}s$. Let $t$ be the lengths of the other two sides. So the area of $WNM$ is $\frac{1}{2}t^2$. By Pythagoras' Theorem $t^2 + t^2 = \left(\frac{1}{2}s\right)^2$. Therefore $\frac{1}{2}t^2 = \frac{1}{16}s^2$. So the area of the triangle $MNY$ is $\frac{1}{4}s^2 - \frac{1}{16}s^2 = \frac{3}{16}s^2$. Hence the ratio of the area of the triangle $MNY$ to the area of the square is 3:16.

**18. B** Let the number of men dancing be $x$ and the number of women dancing be $y$. The information in the question gives the equation $\frac{3}{4}x = \frac{4}{5}y$ i.e. $\frac{x}{y} = \frac{16}{15}$. So $15x = 16y$. As 15 and 16 have no common factors (other than 1), the only solution to this equation for which $x + y < 50$ is $x = 16$ and $y = 15$. Hence the number of people dancing was $\frac{3}{4} \times 16 + \frac{4}{5} \times 15 = 24$.

**19. E** By considering the highest and lowest numbers first, it can be observed that 1 can only be neighbours with 3 and 4, 2 can only be neighbours with 4 and 5, 12 can only be neighbours with 10 and 9 and 11 can only be neighbours with 9 and 8. Hence we have the following chains of neighbours:

3 — 1 — 4 — 2 — 5 and 10 — 12 — 9 — 11 — 8.

This leaves only 6 and 7 to be placed. From the available end-points, 6 can only be joined to 3 and 8 creating the following larger chain of neighbours:

10 — 12 — 9 — 11 — 8 — 6 — 3 — 1 — 4 — 2 — 5.

The circle is then completed by joining 7 to 5 and 10. Hence 6 and 8 must be neighbours.

**20. D** The two-digit square numbers are 16, 25, 36, 49, 64 and 81. Only 1, 4 and 6 are both last digits and first digits of two-digit square numbers so these are the only possibilities for the middle digit of the required integers. Taking each of 1, 4 and 6 in turn gives the list of three-digit integers with the required property as 816, 649, 164 and 364 which have a sum of 1993.

**21. E** Call a story with an odd number of pages an odd story and a story with an even number of pages an even story. The story after an odd story will start on a page number of different parity to the number of the first page of that odd story. In contrast, the story after an even story will start on a page number of the same parity as the number of the first page of that even story. Thus the positions of the even stories have no effect on the parities of the start numbers of the even stories. Hence the odd stories must have 8 starting on odd-numbered pages and 7 starting on even-numbered pages. To maximise the total number of stories starting on an odd page, we must arrange for all the even stories to start on an odd-numbered page which is possible (for example) by positioning them all before the first odd story. This gives the maximum number of stories starting on an odd-numbered page as $15 + 8 = 23$.

**22. B** An equilateral triangle has rotational symmetry of order 3 so only any overall rotation of $360° / 3 = 120°$ (or any multiple of 120°) will leave the triangle occupying the same position. The first few rotations give an overall rotation of $R_1 = 3°$, $R_2 = (3^2 + 3)° = 12°$, $R_3 = (3^3 + 3^2 + 3)° = 39°$ and $R_4 = (3^4 + 3^3 + 3^2 + 3)° = 120°$ which leaves the triangle occupying the same position as the original. Then we have
$R_5 - R_1 = (3^5 + 3^4 + 3^3 + 3^2)° = 3 \times (3^4 + 3^3 + 3^2 + 3)° = 3 \times 120°$. Similarly,
$R_6 - R_2 = (3^6 + 3^5 + 3^4 + 3^3)° = 3^2 \times (3^4 + 3^3 + 3^2 + 3)° = 3^2 \times 120°$,
$R_7 - R_3 = 3^3 \times 120°$, $R_8 - R_4 = 3^4 \times 120°$, $R_9 - R_5 = 3^5 \times 120°$ and so on.
Since all these differences are multiples of 120°, every overall rotation after the fourth gets us back to a position corresponding to one of the first four. Thus only four different positions are possible for the triangle.



**22. C** Consider the six numbers Pablo has chosen. Exactly one pair fails to have the property that the smaller divides the larger. Let $y$ be the smaller number in that pair and note that $y$ is not equal to $N$.

Now consider the other five numbers and put them in numerical order. So each divides $N$ and each divides the next in line. So the second is the product of the first and a number greater than one, the third is the product of the second and a number greater than one, etc. Hence the largest one, $N$, must be the product of at least four numbers greater than 1. What numbers are the product of at least 4 such numbers? The smallest ones are $16\,(= 2 \times 2 \times 2 \times 2)$ and $24\,(= 2 \times 2 \times 2 \times 3)$.

Suppose that $N = 16$. It has only five divisors, 1, 2, 4, 8, 16. So all of these must be in Pablo's set together with $y$, which must be one of the remaining numbers less than 16. No other number under 16 is a factor of 16, so the pair $(y, 16)$ is the pair where the smaller fails to divide the larger. But then $y$ must be a factor or a multiple of 8, neither of which is possible because all factors of 8 and multiples of 8 less than 16 are already on Pablo's list.

Suppose $N = 24$. Its divisors are 1, 2, 3, 4, 6, 8, 12, 24. Then Pablo could choose the set 1, 2, 3, 6, 12, 24 (or alternatively, the set 1, 2, 4, 8, 16, 24). Hence 24 is indeed the least possible choice of $N$.

**23. C** For the 1st, 3rd, 5th, 7th, 9th and 11th jumps, the kangaroo has no choice but must jump to the park (P). On the 13th jump it has no choice but must jump to home (H). On the 2nd, 4th, 6th, 8th, 10th and 12th jumps it can choose either the school (S) or the library (L). Thus the kangaroo has 6 opportunities to choose between 2 options, so has $2^6 = 64$ possible routes.

**24. B** At the start there are five coins showing Heads. In each round two coins are changed. Either two Heads become two Tails, or two Tails become two Heads, or one Tail becomes a Head and one Head becomes a Tail. Any of these possibilities changes the number of Heads by an even number (two less, two more, or no change). Thus there will always be an odd number of Heads, since we started with an odd number. Therefore the coins cannot all show Tails because there would be an even number (zero) of Heads.

It remains to show that the other statements are false.

Since B is true, D and E are obviously false. To deal with A and C, suppose that in each of the first 9 rounds only the first two coins are turned over (one by each player ) . So now those are Tails and the other three are Heads. In the tenth round, if the first two coins are turned over again then all five are Heads and A is shown to be false. On the other hand, if the third and fourth coins are turned over in the tenth round, then four coins are Tails and so C is false.

**25. B** There are 6 choices for the first vertex, and 6 choices for the second vertex giving 36 possible choices. If the first vertex is $H$ then the octagon will not get split into 3 regions. If the first vertex is $G$, then the octagon will be split into 3 regions if the second vertex is $G$, $F$, $E$ or $D$ (4 ways). If the first vertex is $F$ then the octagon will be split into 3 regions if the second vertex is $F$, $E$ or $D$ (3 ways). Continuing in this way, there are 2 ways with $E$ as the first vertex, and one with $D$ as the first vertex. There are no ways if $C$ is the first vertex. Hence there are $4 + 3 + 2 + 1$ choices which split the octagon into three regions. Since there were 36 possible choices, that makes the probability $\frac{10}{36}$; that is $\frac{5}{18}$.

**16. D** We can rearrange the product: $2^{57} \times 3^4 \times 5^{53} = 2^4 \times 3^4 \times (2 \times 5)^{53} = 6^4 \times 10^{53}$. Any power of 6 ends in the digit 6, so this number has last non-zero digit 6 followed by 53 zeroes.

**17. E** Since $MI = MG$, we have $FM + MI = 16$, so $MI = 16 - FM$. By Pythagoras, $FM^2 + FI^2 = MI^2$ so $FM^2 + 16 = (16 - FM)^2$. So $FM^2 + 16 = 256 - 32FM + FM^2$ so $32FM = 240$ and hence $FM = 7.5$ cm. The same argument applies to triangle $H'NI$ giving $H'N = 7.5$ cm and $IN = 8.5$ cm. The areas of triangles $H'NI$ and $FMI$ are both $\frac{1}{2} \times 7.5 \times 4 = 15\,\text{cm}^2$. The area of triangle $MNI = \frac{1}{2} \times 8.5 \times 4 = 17\,\text{cm}^2$, so the area of the pentagon is $15 + 15 + 17 = 47\,\text{cm}^2$.

**18. A** Let $B$ be the length of the train to Brussels, and $u$ be its speed. Let $L$ be the length of the train to Lille and $v$ be its speed. Using speed = distance ÷ time, we get $u = \frac{B}{8}$ and $v = \frac{L}{12}$. When they pass each other, the total length is $B + L$ and the relative speed is $u + v$ so we get $u + v = \frac{B + L}{9}$. Substituting for $u$ and $v$ we get $\frac{B}{8} + \frac{L}{12} = \frac{B + L}{9}$. Multiplying through by 72 gives $9B + 6L = 8B + 8L$ so $B = 2L$. That is, the Brussels train is twice as long as the Lille train.

**19. C** Suppose we fix the hundreds and tens digits, $a$, $b$ say, and consider all the numbers starting with those digits. The sum of the products of their digits will be

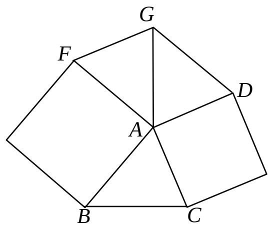
$$a \times b \times (0 + 1 + 2 + \ldots + 9) = a \times b \times 45.$$

Now consider all the numbers starting with hundreds digit $a$. The sum of the products of their digits will be the sum of the expressions $a \times b \times 45$ with $b$ taking all the values 0 to 9; that is

$$a \times (0 + 1 + 2 + \ldots + 9) \times 45 = a \times 45 \times 45.$$

Finally, we let $a$ take all possible values, this time from 1 to 9. The grand total is then $45 \times 45 \times 45 = 45^3$.

**20. B** Note that $\angle BAC + \angle FAD = 180°$ (because the angles at $A$ add up to 360°). And $\angle ADG + \angle FAD = 180°$ (two angles in parallelogram). Hence $\angle ADG = \angle BAC$. Thus triangles $DGA$ and $ABC$ are congruent (as they each have sides of lengths 4 mm and 5 mm which enclose equal angles). The area of triangle $ABC$ is 8 $\text{mm}^2$; so the area of the parallelogram is $2 \times 8 = 16\ \text{mm}^2$.




**21. D** The prime factor decomposition of 2012 is $2^2 \times 503$, so the only possibilities for $m$ are 1 or 2. But if $m = 1$, Anya has written $2012 = 1^1 \times (1^k - k)$, which gives $k = -2011$, contradicting that $k$ is positive. Therefore $m = 2$ and Anya has written $2012 = 2^2 \times (2^k - k)$, so $2^k - k = 503$. Checking powers of 2, it is easy to see that $k = 9$.



**23. C** Cutting the folded ribbon as described will leave three different length strands. There will be two equal length end pieces, three folded strands each double the length of an end piece and four other folded strands all of the same length.

Let the lengths in cm of the three types of strand be $X$, $Y$ and $Z$ respectively where $Y = 2X$. The total length of the ribbon is $2X + 3Y + 4Z = 8X + 4Z$. Now consider the different possibilities for which one of $X$, $Y$ and $Z$ is equal to 4.

Case 1: $X = 4$ which means $Y = 8$ and $Z = 9$ so the total length is $8 \times 4 + 4 \times 9 = 68$ cm.

Case 2: $Y = 4$ which means $X = 2$ and $Z = 9$ so the total length is $8 \times 2 + 4 \times 9 = 52$ cm.

Case 3: $Z = 4$ which means either $X = 9$ and hence $Y = 18$ with a total length of $8 \times 9 + 4 \times 4 = 88$ cm or $Y = 9$ and hence $X = 4.5$ with a total length of $8 \times 4.5 + 4 \times 4 = 52$ cm.

So a total length of 72 cm is impossible.

**24. C** If we add together the sum of the perimeters of the quadrilaterals and the four smaller triangles we get 45 cm. This distance equals twice the sum of the lengths of the three line segments plus the length of the perimeter of the triangle. Hence the sum of the lengths of the line segments is $(45 - 19)/2 = 13$ cm.

**25. A** With the values in each cell as shown

| | | |
|---|---|---|
| $a$ | $b$ | $c$ |
| $d$ | $e$ | $f$ |
| $g$ | $h$ | $i$ |

consider the following fraction

$$\frac{(a \times b \times d \times e) \times (b \times c \times e \times f) \times (d \times e \times g \times h) \times (e \times f \times h \times i)}{(a \times b \times c) \times (d \times e \times f) \times (d \times e \times f) \times (g \times h \times i) \times (b \times e \times h)}.$$

This simplifies to $e$ but, using the rules given for creating the grid, it is also equal to

$$\frac{2 \times 2 \times 2 \times 2}{1 \times 1 \times 1 \times 1 \times 1} = 16.$$

Hence $e = 16$.

*Alternative solution*: With the values in each cell as described above, use the rules given for creating the grid to produce the following equations:

$abde = 2 = bcef$ so $ad = cf$. Also $adg = 1 = cfi$ so $g = i$.
$abde = 2 = degh$ so $ab = gh$. Also $abc = 1 = ghi$ so $c = i$.
$degh = 2 = efhi$ so $dg = fi$. Also $adg = 1 = cfi$ so $a = c$.
Combining these three results gives $a = c = g = i$.

Next, consider the products of the top and bottom rows and the left-hand and right-hand columns, all of which are equal to 1, and deduce that $b = d = f = h = 1/a^2$.

Then, consider the product of the middle row, which is also equal to 1, and deduce that $e = a^4$.

Finally, consider the product of the cells in the top left-hand $2 \times 2$ square and substitute in the formulae obtained for $b$, $d$ and $e$ in terms of $a$ to obtain

$$a \times \frac{1}{a^2} \times \frac{1}{a^2} \times a^4 = 2$$

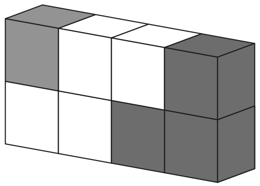
which has solution $a = 2$.

Hence the value of $e$ is $2^4 = 16$.

## Solutions to the European Kangaroo Pink Paper

**1.** **D** $11.11 - 1.111 = 9.999$.

**2.** **C** 



The diagram shows the back eight cubes. So the white piece has shape C.

**3.** **A** Suppose that none of the seven digits is zero. Then the sum must be at least $1 + 1 + 1 + 1 + 1 + 1 + 1 = 7$, which is not true. So one of the digits must be zero. Hence the product of the digits is zero.

**4.** **C** The square $FGHI$ has area $4 \times 4 = 16\,\text{cm}^2$. The triangle $HIJ$ has the same area, and has base 4 cm, so must have height 8 cm. Then the distance of $J$ from the line through $F$ and $G$ is $4 + 8 = 12$ cm.

**5.** **E** Replacing each occurrence of the number 8 by $x$, and simplifying the resulting expressions, we get

A $(x + x - x) \div x = 1$ B $x + (x \div x) - x = 1$ C $x \div (x + x + x) = \frac{1}{3}$

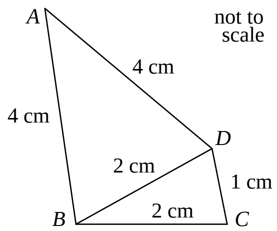
D $x \times (x \div x) \div x = 1$ E $x - (x \div x) + x = 2x - 1$.

The only one that depends on the value of $x$ is E.

**6.** **B** Using Pythagoras' Theorem in triangle $FGH$ we see that its hypotenuse has length 10 cm. Triangle $IJK$ has sides of length half those of $FGH$, so its perimeter is $3 + 4 + 5 = 12$ cm.

**7.** **D** Dividing by $n$ leaves a remainder of 11. This means that $n > 11$ and that $n$ divides exactly into $144 - 11 = 133$ and into $220 - 11 = 209$. The prime factorisation of 133 is $7 \times 19$ and that of 209 is $11 \times 19$. Hence their only common factor, greater than 11, is 19.

**8.** **D** The diagonal of length 2 cm splits the quadrilateral into two isosceles triangles. One of the triangles has sides of length 1 cm and 2 cm, so must have a third side of length 2 cm (it cannot be 1 cm, 1 cm, 2 cm because its three vertices would be collinear). Similarly, the other triangle has sides of length 2 cm, 4 cm and 4 cm (it cannot be 2 cm, 2 cm, 4 cm). Hence the quadrilateral has sides of length 1 cm, 2 cm, 4 cm, 4 cm and perimeter 11 cm.

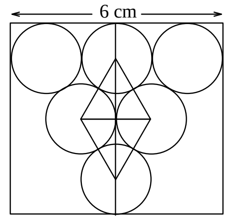



**9.** **C** Let $c$ be Clement's height, $d$ be Dimitri's height and $t$ be the height of the table all in metres. When Clement is on the table, we see that $c + t = d + 0.8$. When Dimitri stands on the table, we see that $d + t = c + 1$. Adding the two equations gives $c + d + 2t = c + d + 1.8$ so $2t = 1.8$ and the height of the table is 0.9 m.



**10.** **B** Let $h$ be the number of heads spun and $t$ the number of tails spun out of thirty spins. Clearly $h + t = 30$. Considering Meinke's sweets, she gains two for every head spun, and loses three for every tail. So at the end of thirty spins she has gained $(2h - 3t)$ sweets, but she ends up with the same number of sweets which means $2h - 3t = 0$. This gives $h = \frac{3}{2}t$. Substituting into $h + t = 30$ we get $\frac{5}{2}t = 30$ and so tails occur 12 times.

**11.** **B** The width of three circles across the top is 6 cm, so each circle has diameter 2 cm and radius 1 cm. The triangles joining the centres of three circles as shown are equilateral with edge lengths 2 cm. By Pythagoras, the heights of the triangles are $\sqrt{2^2 - 1^2} = \sqrt{3}$ cm. By considering the vertical line through the centre of the rectangle, the height of the rectangle is then $1 + \sqrt{3} + \sqrt{3} + 1 = 2\sqrt{3} + 2$.




**12.** **C** Let $T$ be the actual time. Then the clocks show $T \pm 2$, $T \pm 3$, $T \pm 4$, $T \pm 5$ respectively. The earliest and latest times shown differ by nine minutes, which is the maximum possible difference between any of the clocks. It occurs when the last two clocks show either $T + 4$, $T - 5$ or $T - 4$, $T + 5$. Hence the latest shown time of 3:03 is either $T + 5$ or $T + 4$, giving an actual time of 2:58 or 2:59. If it were 2.58 then the clock showing 2.57 would be only one minute wrong, which is not allowed. So 2.59 is the right time.

**13.** **B** Let $H$, $I$, $J$ be the vertices of the triangle, $C$ the centre of the circle, and $K$ the point where the semicircle touches the edge $HI$ as shown. The angle $CKH$ is a right angle because $HI$ is tangent to the circle and so perpendicular to the radius $CK$. The two triangles $HKC$ and $HJI$ are similar since they each have a right angle and they share the angle at $H$. Let $r$ be the radius of the semicircle, then $CK = r$ and $CH = 12 - r$. Then by similar triangles we have

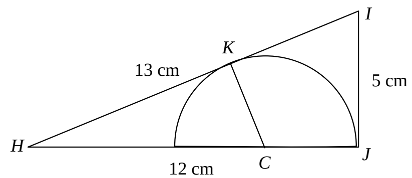
$$\frac{12 - r}{r} = \frac{13}{5}.$$

So $5(12 - r) = 13r$.

Then $60 - 5r = 13r$.

So $18r = 60$

hence $r = \frac{10}{3}$.




**14.** **A** Let the missing entry from the first row be $y$. Since the second and third columns have the same sum, the missing entry in the second column must be $y - 3$. Then the first row adds to $8 + y$, and the third row adds to $4 + y + x$. Since these are the same sum, we must have $x = 4$.

**15.** **E** We write F,G,H for Friedrich, Gottlieb, Hans. Since all the statements are true, we know, from the first one, that F or G came first; so H did not come first. From the second statement we deduce that G did not come second. Hence, from the fourth statement, H did come second. So if G came third then F must have won − which contradicts the third statement. Therefore Gottlieb won and Friedrich came third.